ELSEVIER

Discrete Mathematics 234 (2001) 191–197

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

A necessary and sufficient condition for a graph to be edge tenacious ☆

Zhi-Ping Wang *, Lian-Chang Zhao

Department of Basic Science, Dalian Maritime University, Liaoning, Dalian 116026, People's Republic of China

Received 5 January 1998; revised 14 June 2000; accepted 14 August 2000

Abstract

Piazza, Roberts and Stueckle recently conjectured that every complete n-partite graph is strictly edge-tenacious. In this paper, we establish a necessary and sufficient condition for a graph to be edge-tenacious. We apply these results to prove the conjecture of Piazza et al. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Edge-sets; Edge-tenacious graph; Strictly edge-tenacious graph; Complete n-partite graph

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let S be a subset of $E(G)$. Denote by $\omega(G-S)$ the number of (connected) components of $G-S$, by $\tau(G-S)$ the order (number of vertices) of a largest component of $G-S$.

The score of S is defined as $\mathrm{sc}(S)=[|S|+\tau(G-S)]/[\omega(G-S)]$. Formally, the edge-tenacity of a graph G is defined as $T'(G)=\min\{\mathrm{sc}(S)\}$, where the minimum is taken over all edge-sets S of G. Let $T(G)=\min\{\mathrm{sc}(S)\}$, where the minimum is taken over all edge-sets $S\neq E$ of G. A subset $S\neq E$ of $E(G)$ is said to be a T-set of G if $T(G)=\mathrm{sc}(S)$. Note that if G is disconnected, then the set S may be empty. Throughout this paper, we use ω and τ to represent $\omega(G-S)$ and $\tau(G-S)$, respectively, when G and S are clear from the context. We also use p and q to represent the number of vertices (order) and the number of edges (size), respectively, of a graph. The edge-connectivity of G will be denoted $\lambda=\lambda(G)$. Definitions and notation not otherwise defined here can be found in [1].

☆ This research was supported by National Natural Science Foundation of China.
* Corresponding author.
E-mail address: lczhao@dlmu.edu.cn (L.-C. Zhao).

PII: S0012-365X(00)00400-3

A graph G is called edge-tenacious if $T'(G)=\mathrm{sc}(E(G))$. A graph is called strictly edge-tenacious if E is the unique set whose score equals $T'(G)$. Edge-tenacious graphs can be considered very stable, because to minimize the ratio of cost to reward, an attacker needs to destroy all of the edges. Thus, attacks tend to be 'expensive' and so the networks are relatively invulnerable. Many network topologies used to design highly reliable computer, communication, and transportation networks are edge-tenacious (see [2]).

The area of graph vulnerability concerns the question of how much communication in a network is disrupted by the deletion of edges from the graph. The most fundamental measure of graph vulnerability of a connected graph is the edge-connectivity of the graph. The difficulty with the edge-connectivity is that it does not take into account what remains after the graph is disconnected. Consequently, another parameter has recently been introduced that attempts to cope with this difficulty, such as edge-toughness [3].

In edge-toughness, the 'cost' to an 'attacker' of destroying S is the size of S and the 'reward' is measured by the number of components left after destroying S (since creating more components makes it harder to reconnect a network). In edge-tenacity, the 'cost' also takes into account the size of the largest remaining component, since a larger remaining component means the 'attack' was not quite as successful.

2. Preliminary results

We first state the following result, which will play a key role in this paper.

Theorem 1. *Let G be a graph, and let S be a T-set of G. Suppose that H is the union of all nontrivial components of $G-S$. Then*

$$\frac{q+1}{p}\leqslant \mathrm{sc}(S)$$

if and only if

$$\frac{q'-\tau'+1}{p'-\omega'}\leqslant\frac{q+1}{p}$$

$(p'=|V(H)|,\ q'=|E(H)|,\ \omega'=\omega(H),\ \tau'=\tau(H))$.

To prove Theorem 1, we shall need the following lemma.

Lemma 1. *Let a,b,x,y be positive integers, and let $x>a, y>b$. Then*

$$\frac{x}{y}\leqslant\frac{x-a}{y-b}$$

if and only if

$$\frac{a}{b}\leqslant\frac{x}{y}.$$

The proof of the result follows from basic algebraic manipulations.

Proof of Theorem 1. Suppose

$$\frac{q+1}{p} \leqslant \mathrm{sc}(S)$$

for a T-set S of G. Let H be the union of all nontrivial components of $G-S$ with p' vertices and q' edges. If $p'=p$, $\omega'=1$, $\tau'=p$, then it is clear that

$$\frac{q'-\tau'+1}{p'-\omega'} \leqslant \frac{q+1}{p}.$$

Otherwise, let x be the cardinality of T-set $S=E(G)-E(H)$, which separates G into H and isolated vertices. Then $q=q'+x$ and $\omega(G-S)=p-p'+\omega'$, $\tau(G-S)=\tau(H)=\tau'$. Therefore, we have

$$\frac{q+1}{p} \leqslant \mathrm{sc}(S) = \frac{q-q'+\tau'}{p-p'+\omega'}.$$

By Lemma 1, we get

$$\frac{q'-\tau'+1}{p'-\omega'} \leqslant \frac{q+1}{p}$$

as required.

Conversely, let S be any T-set separating G into $\omega(G-S)$ components $H_1, H_2, \ldots, H_t$, where $t=\omega(G-S)$. Suppose that $H_1, H_2, \ldots, H_{t'} (1 \leqslant t' \leqslant t)$ are nontrivial components. Let H be the union of $H_1, H_2, \ldots, H_{t'}$. Then

$$\frac{q-|S|-\tau(G-S)+1}{p-(t-t')-(\omega(G-S)-(t-t'))} \leqslant \frac{q+1}{p}.$$

By Lemma 1, we get

$$\frac{q+1}{p} \leqslant \frac{\tau(G-S)+|S|}{\omega(G-S)}.$$

Therefore,

$$\frac{q+1}{p} \leqslant \mathrm{sc}(S)$$

as required.

From Theorem 1, we have:

Remark. Let G be a graph, and let S be a T-set of G. Suppose that H is the union of all nontrivial components of $G-S$. If

$$\frac{q'-\tau'+1}{p'-\omega'} < \frac{q+1}{p},$$

then

$$\frac{q+1}{p} < \mathrm{sc}(S)\ (p' = |V(H)|,\ q' = |E(H)|,\ \omega' = \omega(H),\ \tau' = \tau(H)).$$

As a corollary, we have:

Corollary 1. *Let G be a graph, and let S be a T-set of G. Assume that $G - S$ have at most ω' nontrivial components. If*

$$\frac{q_i}{p_i - 1} < \frac{q+1}{p} + \frac{1}{\omega'},$$

for any nontrivial component H_i of $G - S$ ($p_i = |V(H_i)|,\ q_i = |E(H_i)|,\ i = 1, 2, \ldots, k;\ k \leqslant \omega'$), then G is strictly edge-tenacious.

Proof. Suppose that $G - S$ has k ($1 \leqslant k \leqslant \omega'$) nontrivial components. Let

$$\tau' = \tau(G - S), \quad p' = \sum_{i=1}^{k} p_i, \quad q' = \sum_{i=1}^{k} q_i.$$

If

$$\frac{q_i}{p_i - 1} < \frac{q+1}{p} + \frac{1}{\omega'},$$

then

$$\frac{q_i}{p_i - 1} < \frac{q+1}{p} + \frac{1}{k}.$$

Thus,

$$\frac{q_i - (\tau' - 1)/k}{p_i - 1} \leqslant \frac{q_i - (p_i - 1)/k}{p_i - 1} = \frac{q_i}{p_i - 1} - \frac{1}{k} < \frac{q+1}{p}.$$

Therefore,

$$q' - (\tau' - 1) < \frac{q+1}{p}(p' - k).$$

Thus,

$$\frac{q' - \tau' + 1}{p' - k} < \frac{q+1}{p}.$$

By Theorem 1, we have

$$\frac{q+1}{p} < \mathrm{sc}(S).$$

Hence, G is strictly edge-tenacious.

The following theorem gives the possible relationships between scores for two arbitrary subsets of E. This result gives us a very useful tool for deciding whether a set is a T-set.

Theorem 2 (Piazza et al. [2]). *Let S and S' be subsets of E such that $|S'| - |S| = a, \omega(G-S') - \omega(G-S) = b$ and $\tau(G-S') - \tau(G-S) = -c$. Then,*

(i) $\mathrm{sc}(S') < \mathrm{sc}(S)$ *if and only if* $[(a-c)/b] < \mathrm{sc}(S)$,
(ii) $\mathrm{sc}(S') = \mathrm{sc}(S)$ *if and only if* $[(a-c)/b] = \mathrm{sc}(S)$,
and
(iii) $\mathrm{sc}(S') > \mathrm{sc}(S)$ *if and only if* $[(a-c)/b] > \mathrm{sc}(S)$.

Recall that edge-toughness of a graph G is

$$\tau_1(G) = \min\left[\frac{|S|}{\omega(G-S)-1}\right],$$

where the minimum is taken over every edge-cutset S that separates G into $\omega(G-S)$ components. We shall also need the following result.

Theorem 3 (Peng et al. [3]). *Let G be a graph, and let $s = q/(p-1)$. Then $\tau_1(G) = s$ if and only if $|E(H)| \leqslant s(|V(H)|-1)$, for every subgraph H of G.*

Corollary 2. *Let G be a graph, and let S be a T-set of G. Assume that $G-S$ have at most two nontrivial components. If $\tau_1(G) = q/(p-1)$, then G is strictly edge-tenacious.*

Proof. Since

$$q \leqslant \tfrac{1}{2}p(p-1) < \tfrac{1}{2}(p+2)(p-1),$$

thus,

$$\frac{q}{p-1} \leqslant \frac{q+1}{p} + \frac{1}{2}.$$

By Theorem 3, we get

$$\frac{q_i}{p_i-1} \leqslant \frac{q}{p-1} \leqslant \frac{q+1}{p} + \frac{1}{2},$$

for any nontrivial component H_i of $G-S(p_i = |V(H_i)|,\ q_i = |E(H_i)|,\ i = 1, 2)$. Therefore, by Corollary 1, we have, G is strictly edge-tenacious.

Remark. Assume that $G-S$ has at most one nontrivial component. If $\tau_1(G) = q/(p-1)$, then G is strictly edge-tenacious.

3. Complete *n*-partite graphs

Piazza, Roberts and Stueckle recently conjectured that every complete n-partite graph is strictly edge-tenacious. The aim of this section is to give a proof of the conjecture of Piazza et al.

Theorem 4. *Let $G=K(m_1,m_2,\dots,m_n)$ be a complete n-partite graph, where $n\geqslant 2$ and $m_n\geqslant m_{n-1}\geqslant\cdots\geqslant m_1\geqslant 1$. Then G is strictly edge-tenacious.*

We shall use Corollary 2 to prove the theorem above. To do so we need the following lemmas, which are Theorem 3 of Piazza et al. [2] and Theorem 3.1 of Peng et al. [3], respectively.

Lemma 2. *Suppose that G has edge-connectivity λ. Then $T'(G)\geqslant\lambda/2+1/p$.*

Lemma 3. *If G is a complete n-partite graph with p vertices and q edges, then $\tau_1(G)=q/(p-1)$.*

From Theorem 2, as in the proof of Corollary 2 of Piazza et al. [2], we have

Lemma 4. *If S is a T-set and C is a nontrivial component of $G-S$, then $\lambda(C)\geqslant T(G)$.*

Proof of Theorem 4. Let S be a T-set. By Lemma 4, each of nontrivial components of $G-S$ must have edge-connectivity at least $T(G)$. Assume that $G-S$ contains at least two nontrivial components H_1 and H_2. By Lemma 2, we have

$$\lambda(H_1)\geqslant T(G)\geqslant T'(G)\geqslant\frac{\sum_{i=1}^{n-1}m_i}{2}+\frac{1}{\sum_{i=1}^{n}m_i}>\frac{\sum_{i=1}^{n-1}m_i}{2},$$

$$\lambda(H_2)\geqslant T(G)\geqslant T'(G)\geqslant\frac{\sum_{i=1}^{n-1}m_i}{2}+\frac{1}{\sum_{i=1}^{n}m_i}>\frac{\sum_{i=1}^{n-1}m_i}{2}.$$

Let $V_1,V_2,\dots,V_n$ be the subsets of vertices in the complete n-partite graph G, where each V_i induces isolated vertices. Let $A_1,A_2,\dots,A_n$ be the subsets of vertices in H_1 such that $A_i\subset V_i$ (possibly $A_i=\emptyset$). Similarly, define $B_1,B_2,\dots,B_n$ for H_2. Denote $a_i=|A_i|$ and $b_i=|B_i|$. Consider the graph H induced by $A_1\cup B_1$, $A_2\cup B_2$, $\dots$, $A_n\cup B_n$. Then

$$\lambda(H_1)=\sum_{i=1}^{n}a_i-\max_{1\leqslant i\leqslant n}a_i,\quad \lambda(H_2)=\sum_{i=1}^{n}b_i-\max_{1\leqslant i\leqslant n}b_i$$

and

$$\lambda(H)=\sum_{i=1}^{n}(a_i+b_i)-\max_{1\leqslant i\leqslant n}(a_i+b_i).$$

From this, we have $\lambda(H)\geqslant\lambda(H_1)+\lambda(H_2)$, since

$$\max_{1\leqslant i\leqslant n}a_i+\max_{1\leqslant i\leqslant n}b_i\geqslant\max_{1\leqslant i\leqslant n}(a_i+b_i)$$

is obvious.

Thus,

$$\lambda(H) \geqslant \lambda(H_1) + \lambda(H_2) > \sum_{i=1}^{n-1} m_i.$$

Since $\lambda(G) \geqslant \lambda(H)$, thus,

$$\lambda(G) > \sum_{i=1}^{n-1} m_i,$$

a contradiction. It follows that $G-S$ has at most one nontrivial component. By Lemma 3 and Corollary 2, we have, G is strictly edge-tenacious.

Acknowledgements

The authors would like to thank the anonymous referees for their constructive suggestions and critical comments which resulted in a much improved paper.

References

[1] G. Chartrand, L. Lesniak, Graphs and Digraphs, 2nd Edition, Wadsworth, Monterey, CA, 1986.
[2] B.L. Piazza, F.S. Roberts, S.K. Stueckle, Edge-tenacious networks, Networks 25 (1995) 7–17.
[3] Y.H. Peng, T.S. Tay, On the edge-toughness of a graph II, J. Graph Theory 17 (1993) 233–246.